This Week in SM311P:1001: Homework, etc.

Homework must be submitted stapled in assignment groupings.

Always attempt to complete the readings before class. You are responsible for reading 10 pages past the current lecture. You may not understand the material completely, but you must read it prior to lecture.

** Problems to submit on the date listed: **

Week of 13 Oct

Monday		
Wednesday:	Submit	LT: 1,2,4,5 Read problems 6-9 and 11
Friday:	Submit	LT: 10,12, 13, 16, 17

Week of 20 Oct HOUR EXAM II on Friday 24 Oct

TERMS & PRINCIPLES TO TREASURE:

- matrix
- transpose
- cofactor
- Cramer's Rule
- Hermitian conjugate
- permutation symbol
- inverse

MD: Introduction to Matrices and Determinants
VS: => Vector Spaces Handout Problem
EV: Eigenvalue Problem
RR: Row Reduction

Hints: 1 is not very difficult. 5 is easy ! 4 can be based on the example in the handout, but it is not exactly the same. Use the expressions for matrix multiplications as sums and the properties of the matrices involved for 2, 12 and 16. Read carefully to find the relation of the transformations required to diagonalize matrices and their eigenvectors.

1.) Consider the case of a primed frame rotated by θ about the z-axis relative to the unprimed frame and a double-primed frame rotated by ϕ about the z-axis relative to the primed frame.

$$\begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \begin{bmatrix} a_1'' \\ a_2'' \\ a_3'' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}$$

Apply the transformations in sequence to show that:

$$\begin{bmatrix} a_1'' \\ a_2'' \\ a_3'' \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) & 0 \\ -\sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In sequence means that each subsequent operation is represented by multiplication of the left side

2.) Show that a similarity transformation of a real symmetric matrix with a real orthogonal matrix and its inverse results in a real symmetric matrix.

3.) Show that a similarity transformation of a Hermitian matrix with a unitary matrix and its inverse results in a Hermitian matrix.

4.) Given the equation for an ellipse in standard form with major and minor axes parallel to the x and y axes, transform to the view from a frame rotated by θ about the z-axis. Show that:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) x'^2 + \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \sin(2\theta)\, x' y' + \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right) y'^2 = 1$$

which gives: $A = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$; $B = \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)$; $C = \left(\frac{1}{2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \sin(2\theta)$ in the expression for an ellipse $A\, x'^2 + 2\, C\, x' y' + B\, y'^2 = 1$

Is the angle θ used here the same as the one used in the discussion of the rotated ellipse in this linear transformations section ?

5.) Considering $\tilde{\tilde{\alpha}}$ as a matrix, what is the relation between $|\tilde{\tilde{\alpha}}|$ and $|\tilde{\tilde{\alpha}}'|$ where $|\tilde{\tilde{\alpha}}|$ is the determinant of the polarizability matrix ? The determinant of a matrix is a scalar as is the magnitude of a vector. How do we expect scalars to transform ?

6.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the image of $\mathbb{V}$ in $\mathbb{W}$ under the mapping is a vector space (the full space $\mathbb{W}$ or a subspace of $\mathbb{W}$). Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field that is also the field for the linear map. Note that most of the properties of addition and multiplication are not issues. Why ? What are the issues that must be addressed?

7.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the kernel of the mapping is a subspace of $\mathbb{V}$. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field which is also the field for the linear map. Note that most of the properties of addition and multiplication are not issues. Why ? What are the issues that must be addressed ?

8.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the set of images of a basis set $\mathbb{V}$ in $\mathbb{W}$ under the mapping is a spanning set for the image of $\mathbb{V}$ in $\mathbb{W}$. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field which is also the field for the linear map.

9.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the dimension of $\mathbb{V}$ minus the dimension of the kernel of the mapping is equal to the dimension of the image of $\mathbb{V}$ in $\mathbb{W}$ under the mapping. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field which is also the field for the linear map.

10.) In the moment of inertia example, the claim is made that $\mathbb{I}$ could be computed by finding the transformed coordinates of the particles and exercising the formula:

$I'_{ij} = \sum_{masses\ \alpha} m_\alpha \left[r'^2_\alpha\, \delta_{ij} - x'_{\alpha i}\, x'_{\alpha j} \right]$ Please do so. Use the transformation equations:

(remember $\theta = -\sin^{-1}(3/5)$)

$$x' = x\cos\theta + y\sin\theta, \quad y' = x\left(-\sin\theta\right) + y\cos\theta, \quad z' = z$$

to compute the coordinates of the masses in the rotated frame. Compute the moment of inertia tensor as observed in the rotated frame using the transformed coordinates. Masses are neither vectors nor (rank two) tensors. What is the transformation type for masses and inner products of vectors? How do the values of the masses of each object compare as determined by the original observer and by the observer in the rotated frame ?

11.) Referring to the Eulerian angles, compute the rotation corresponding to $\varphi = \pi/2$; $\theta = \pi/4$; $\psi = 0$ and to $\varphi = 0$; $\theta = \pi/4$; $\psi = \pi/2$. How are the two rotation sequences related? Are the resulting overall rotations equivalent ? What statement can you make that summarizes your observations ?

$$\lambda(\phi,\theta,\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12.) It has been demonstrated that the coordinate dyadic transforms as a rank two tensor.

$$\mathbb{d}' = \lambda\, \mathbb{d}\, \lambda^{-1} \quad \text{where } [\mathbb{d}]_{ij} \text{ is } x_i\, x_j$$

$$\text{Recall } x_i' = \sum_j \lambda_{ij}\, x_j \text{ so:}$$

$$d_{ij}' = x_i' x_j' = \sum_{m,\ell} \lambda_{im}\, x_m \lambda_{j\ell} x_\ell = \sum_{m,\ell} \lambda_{im}\, x_m x_\ell \lambda_{j\ell}$$

$$= \sum_{m,\ell} \lambda_{im}\, x_m x_\ell \lambda^t_{\ell j} = \sum_{m,\ell} \lambda_{im}\, x_m x_\ell \lambda^{-1}_{\ell j} = \sum_{m,\ell} \lambda_{im}\, d_{m\ell} \lambda^{-1}_{\ell j}$$

a.) Show that the Kronecker delta represents the elements of a rank two tensor in the sense that the elements transform according to the similarity transformation rule.

b.) Conclude that $I_{ij} = \sum_{masses\ \alpha} m_\alpha \left[r_\alpha^2\, \delta_{ij} - x_{\alpha i} x_{\alpha j} \right]$ transforms as a rank two tensor given that m_α and r_α^2 are scalars.

13.) The matrix $\mathbb{A} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ has eigenvalues {1, 3, 3}. Pick linearly independent eigenvectors for these eigenvalues and use them to develop an orthogonal transformation matrix λ that diagonalizes $\mathbb{A}$. Apply the resulting similarity transformation to demonstrate that $\mathbb{A}$ is transformed to a diagonal form with the eigenvalues as the diagonal elements. $\mathbb{A}' = \lambda\, \mathbb{A}\, \lambda^{-1}$; $\mathbb{A}'$ diagonal.

14.) Consider the space of functions of the form

$$f(x) = a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + a_4 \sin(x) + a_3 \sin(2x) + a_6 \sin(3x)$$

with function vectors:

$$|f(x)\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \qquad \langle f(x)| = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

Show that the matrices for the linear operations $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ have representations as (the • is read as 'the operator matrix for'):

$$\frac{\hat{d}}{dx} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\hat{d}^2}{dx^2} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}$$

a.) Evaluate the result of each operator on $f(x) = \cos(2x) - \sin(3x)$ which is

$$| f(x)\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

b.) Compute the matrix for $\frac{\hat{d}}{dx}$ followed by $\frac{\hat{d}}{dx}$.

15.) Consider the space of functions of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with function vectors similar to those for the previous problem, but with index numbers running from zero to infinity.

a.) Show that the matrices for the linear operations $\frac{d}{dx}$ and x have representations as (the • is read as 'the operator matrix for'):

$$\frac{\hat{d}}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 3 & \ldots \\ 0 & 0 & 0 & 0 & 4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \quad \text{and} \quad \hat{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & 1 & \ldots \end{bmatrix}$$

That is: the operator matrix for the derivative with respect to x has zero elements with the exception of the counting numbers on the first super-diagonal. The matrix for multiplication by x has zero elements with the exception of the ones on the first sub-diagonal. Recall that rows and columns are numbered from ZERO.

b.) Show that the matrix for $\frac{d}{dx}\, x - x \frac{d}{dx}$ is the matrix all zero elements excepts for ones on the diagonal. The difference of operators acting in reversed order is called a commutator. The result above stated that $\frac{d}{dx}$ and x do not commute.

c.) Show that $\frac{d}{dx}\left(x \; f(x)\right) - x\,\frac{d}{dx}\,f(x) = f(x) \quad f(x)$. Equivalently, $\frac{d}{dx}\;x - x\,\frac{d}{dx}$ is the identity operator for all differentiable functions.

16. The trace of a matrix is defined to be the sum of its diagonal elements.

$\mathrm{Tr}[\,\mathbb{M}\,] = \sum_{i\,j} \delta_{i\,j}\, m_{i\,j}$ Show that he trace of a matrix is invariant under similarity transformation.

17. A standard problem in quantum mechanics* examines the energies of two states that are initially degenerate (have the same energy) when a small additional interaction couples the two states. It begins with the matrix:

$$\begin{bmatrix} E_0 & \Delta \\ \Delta & E_0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors for this matrix. Find the transformation matrix that diagonalizes the matrix. It happens that the matrix for the transformation is orthogonal. * One embodiment is the energies of the 2S and $2P_z$ state of a hydrogen atom placed in a uniform external electric field $\mathcal{E}\,\hat{k}$. See pages 252-255 of the text by Schiff.

18. An interesting variant of the previous problem studies the matrix:

$$\begin{bmatrix} E_0 + \varepsilon & \Delta \\ \Delta & E_0 - \varepsilon \end{bmatrix}$$

Examine the eigenvectors and eigenvalues for this problem. Compare and contrast the results for the limits $\Delta << \varepsilon$ and $\varepsilon << \Delta$. This matrix is related to a problem in which two nearly degenerate levels are coupled by a perturbation.

Eigenvalues: $E_0 \pm \sqrt{\Delta^2 + \varepsilon^2}$